Math 4650 - Homework # 5
Continuity

1. For each of the following, use the ϵ-δ definition of continuity to prove that the given f is continuous at the given a.

   (a) $f(x) = 2x + 1$ at any $a \in \mathbb{R}$

   (b) $f(x) = x^4$ at any $a \in \mathbb{R}$
   [Hint: Use $x^4 - a^4 = (x^2 + a^2)(x^2 - a^2) = (x^2 + a^2)(x - a)(x + a)$]

   (c) $f(x) = x^2 + x$ at any $a \in \mathbb{R}$

   (d) $f(x) = \dfrac{1}{x^2}$ at $a > 0$.

---

2. Let $f : D \to \mathbb{R}$ where $D \subseteq \mathbb{R}$ and $a \in D$. Prove that f is continuous at a if and only if $\lim_{n\to\infty} f(x_n) = f(a)$ for every sequence (x_n) contained in D with $x_n \to a$.

---

3. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous at $a \in \mathbb{R}$. Prove that if $f(a) > 0$, then there exists $\delta > 0$ such that $f(x) > 0$ for all x where $|x - a| < \delta$.

---

4. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous on all of $\mathbb{R}$. Let $S = \{x \mid f(x) = 0\}$ be the set where f is equal to zero. Suppose that S is not the empty set. Prove that if (x_n) is a sequence of points contained in S and $x_n \to L$, then $f(L) = 0$.

---

5. Suppose that $f : D \to \mathbb{R}$ is continuous on $D \subseteq \mathbb{R}$. Further suppose that $g : A \to \mathbb{R}$ where $A \subseteq \mathbb{R}$ and that the range of g is contained in D. Suppose that $a \in A$ is a limit point of A and $\lim_{x\to a} g(x) = L$ with $L \in D$. Prove that

$$\lim_{x\to a} f(g(x)) = f\left(\lim_{x\to a} g(x)\right) = f(L)$$

---

6. Prove the following.

   (a) Prove that $f(x) = x$ is continuous for all $d \in \mathbb{R}$.

   (b) Let α be a constant real number. Prove that the constant function $f(x) = \alpha$ is continuous for all $d \in \mathbb{R}$.

   (c) Prove that polynomials are continuous for all $d \in \mathbb{R}$.